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# An Attitude Compensation Technique for a MEMS Motion Sensor Based Digital Writing Instrument

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***Abstract*—A MAG-μIMU which is based on MEMS gyroscopes, accelerometers, and magnetometers is developed for real-time estimation of human hand motions. Appropriate filtering, transformation and sensor fusion techniques are combined in the Ubiquitous Digital Writing Instrument to record handwriting on any surface. In this paper, we discuss the design of an extended Kalman filter based on MAG-μIMU (Micro Inertial Measurement Unit with Magnetometers) for real-time attitude tracking. The filter utilizes the gyroscope propagation for transient updates and correction by reference field sensors, such as gravity sensors, magnetometers or star trackers. A process model is derived to separate sensor bias and to minimize wide-band noise. The attitude calculation is based on quaternion which, when compared to Euler angles, has no singularity problem. Testing with synthetic data and actual sensor data proved the filter will converge and accurately track the attitude of a rigid body. Our goal is to implement this algorithm for motion recognition of a 3D ubiquitous digital pen.**

***Keywords-MEMS; MAG-μIMU; Human Motion Sensing; extended Kalman filter ; Digital Writing System; Wireless Sensing***

## I. INTRODUCTION

"Electronic Whiteboard" and "Digital Pen" are new paradigms in the office automation industry that may someday completely replace the computer keyboard, which is still the preferred alphanumeric human-to-computer input device. An Ubiquitous Digital Writing Instrument has been developed by our group to capture and record human handwriting or drawing motions in real-time based on a MEMS Micro Inertial Measurement Unit (μIMU) [1].

In inertial kinematic theory, accurate attitude is fundamental to determine and to keep track of the position of a rigid body in motion. However, due to integrations and triangle functions, the bias error and random walk noise from attitude sensors will be accumulated, magnified and cause nonlinear errors leading to distortions in position tracking.

Attitude tracking is widely used in navigation, robotics, and virtual reality. The problem is addressed by the Attitude Heading Reference System (AHRS). The AHRS utilizes gyroscope propagation for transient updates and correction by reference field sensors. However, classically, the performance is ensured by extremely accurate sensors and hardware filters. Due to its expensive cost and large system size, the AHRS has been limited in application.

New reliable attitude tracking systems are developed based on low cost inertial sensors and the Global Positioning System (GPS). For feedback correction, Euler angles are derived from GPS to represent spatial rotation and a Kalman filter is implemented to fuse with the attitude propagation. But GPS signals are not available for indoor applications and the GPS attitude has a resolution limitation for small objects [2].

With MEMS sensing technology, the inertial sensors are low in price and accurate even at small sizes. In many applications under motionless situations such as for Unmanned Aerial Vehicles (UAV) ground control, the MEMS accelerometers work reliable as gravity sensors. Euler angles can be derived directly [3]. However, the pitch attitude along the gravity axis cannot be determined. Further, during motion, inertial accelerations will interfere with the gravitational accelerations, which then cannot be trusted for attitude.

Magnetometers experience no such crosstalk disturbance in both situations. However, following the same approach, attitude ambiguity occurs along the magnetic field direction. Euler Angles cannot be derived directly. Further, the Earth magnetic field is overlapped by random noise from electromagnetic interference (EMI).

Thus, an extended Kalman filter is designed to integrate the gyroscopes and magnetometers in the MAG-μIMU. For the digital writing instrument application, the MAG-μIMU is affixed on a commercially available marker. During hand motions, the filter tracks the real-time attitude of the pen with sensor bias separation and sensor random noise minimization. The attitude calculation is totally based on quaternion which computes faster and has no singularity problem compared to Euler angles. This filter also applies to other reference field sensors for feedback, such as accelerometers or star trackers.

This paper is organized as follows. In Section 2, the design of the MAG-μIMU attitude system is introduced, including the hardware structure and software architecture. We will describe the design of the extended Kalman filter in Section 3. Simulation examples and experiment results will be discussed in Section 4. Finally, we present conclusions and proposed future improvements in the last section.

## II. ARCHITECTRU OF MAG-μIMU

### A. *Hardware for Attitude filter*

Fig. 1 illustrates the block diagram of a wireless MAG-μIMU with the real-time attitude filter system. The system can be divided into two parts. The first part is the hardware for the wireless sensing unit. The other part is the software for data access and 3D rotation sensing algorithms.

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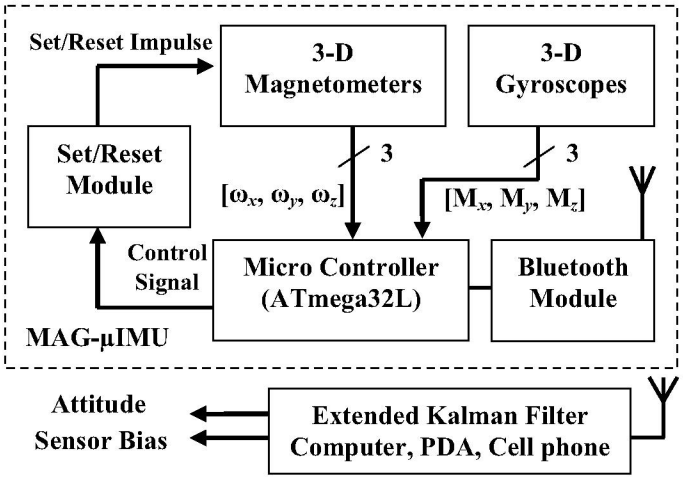


Figure 1. Wireless MAG-µIMU block diagram

The MAG-µIMU is developed for a wireless digital writing instrument and used to record human handwriting. The MAG-µIMU is a hybrid sensing system with inertial sensors and magnetometers. The 'µIMU' integrates the 3D accelerometers and 3D gyroscopes with strapdown installation. [1]. The 3D magnetometers, 'MAG' sensors, are added to measure the Earth magnetic field. The sensor unit is affixed on a commercially available marker to measure the inertial and magnetic information in the pen's body frame.

The output signals of the accelerometers [$A_X$, $A_Y$, $A_Z$] and the gyroscopes [$\omega_X$, $\omega_Y$, $\omega_Z$], which are the body frame accelerations and the roll, pitch, yaw angular rates, respectively, are measured directly with an Atmega32L A/D converter microcontroller. The serial Bluetooth transceiver is implemented via a USART connection with the MCU for wireless communications. The digital sample rate of the sensor unit is 200 Hz and the transmit baud rate is 57.6 Kbps, which ensures rapid reaction to human handwriting.

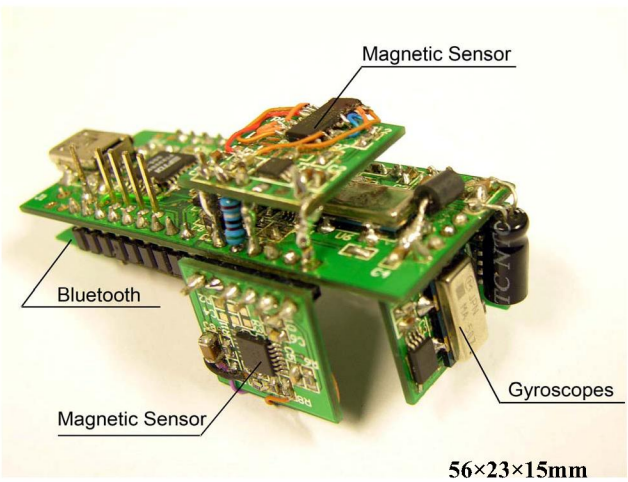


Figure 2. The Prototype of the MAG-µIMU with Bluetooth Module

Fig. 2 shows the prototype MAG-µIMU with the Bluetooth module for wireless connection. The sensor system utilizes four-layer printed circuit board techniques for noise reduction. The dimensions are within 56×23×15*mm*.

### *B. Handwriting Recording for a Digital Writing Instrument*

Fig. 3 illustrates the MAG-µIMU sensor structure of the digital writing system for position tracking.

According to the strapdown kinematics theory [4], the body frame accelerations are transformed to the Earth frame by a Direct Cosine Matrix (DCM). After compensating for the gravitational and rotational accelerations, the translation accelerations integrate into 3D trajectories in space. Thus any 2D human handwriting is recorded in real time if the pen touches the white board plane.

$$\vec{A}_b = \vec{A}_{IMU} - \vec{A}_{Rotation}(\omega_{roll}, \omega_{pitch}, \omega_{pitch}, L) \quad (1)$$

$$\dot{\vec{V}}_e = DCM(q)_b^e \vec{A}_b - \vec{G} \quad (2)$$

$$\vec{P}_e = \iint \dot{\vec{V}}_e \quad (3)$$

Where $\vec{A}_{IMU}$ are the body frame accelerations: [$A_X$, $A_Y$, $A_Z$]. $q$ is the quaternion representing the pen attitude. $\vec{G}$ is the gravity vector: $[0,0,-g]$.

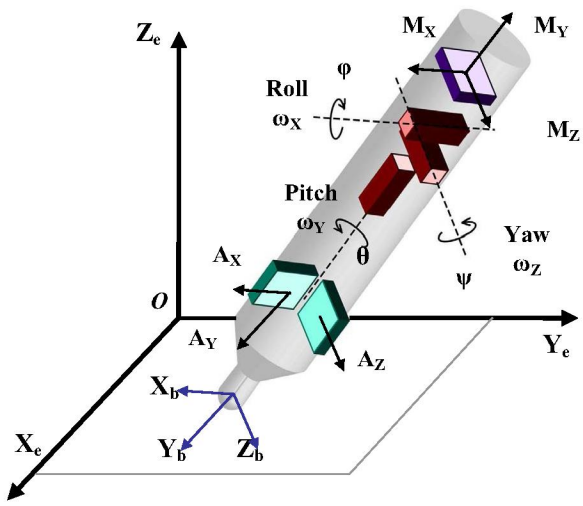


Figure 3. The MAG-µIMU System Structure for a wireless digital writing instrument.

## III. ATTITUDE IN QUATERNION

There are many methods to represents angular position from one reference frame to another reference frame. Each has some advantages and disadvantages. Euler angles and quaternions are commonly used. The Euler angles define one orientation motion in three successive rotations: roll, pitch, and yaw about the X, Y, and Z axis respectively. However, the calculations of Euler angles are trigonometric functions which require costly computation time and will induce nonlinear error. This problem will be manifest since the rotation matrix is singular when the pitch angle is equal to or even close to 90 degrees [5].

The quaternion representation of rigid body rotations leads to convenient expressions [6]. A quaternion is four-dimensional complex numbers.

$$q = q_1 i + q_2 j + q_3 k + q_4 = \vec{q} + q_0 \tag{4}$$

Where $q_1$, $q_2$, $q_3$, and $q_4$ are the real numbers and $i$, $j$, and $k$ are the unit vectors directed along the $X$, $Y$, and $Z$ axis respectively. After normalization for $q$, a unit quaternion $n$ can be expressed in the vector and angle form:

$$n = \frac{q}{\|\vec{q}, q_0\|} = (e_1 i + e_2 j + e_3 k)\sin\frac{\theta}{2} + \cos\frac{\theta}{2} \tag{5}$$

The orientation motion of vector $u$ from the reference frame to the destination frame can be expressed by rotation about the vector $\vec{e}$ for $\theta$ from the quaternion $n$. The rotation is performed through quaternion multiplication.

$$\vec{v} = n \otimes \vec{u} \otimes n^* \tag{6}$$

Where $n^*$ is the quaternion conjugate which is defined as:

$$n^* = -n_1 i - n_2 j - n_3 k + n_4 \tag{7}$$

## IV. Attitude Extended Kalman Filter

The extended Kalman filter consists of two stages. In the time update stage, the quaternion increment by the gyroscopes will propagate the attitude in time. In the measurement update stage, the difference between the estimated and the measured Earth magnetic vector is implemented as feedback to correct the propagation error [7].

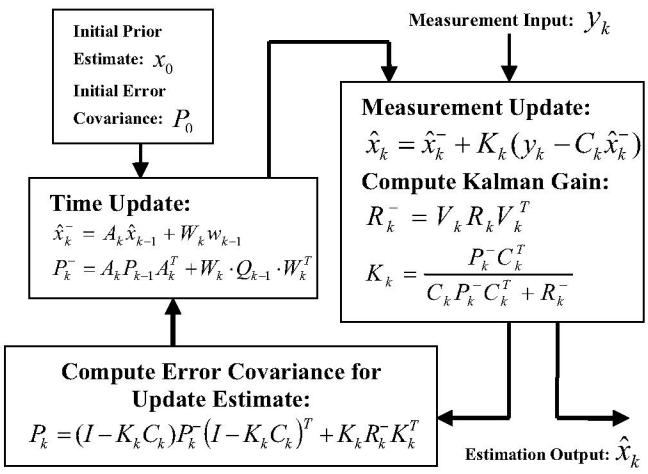


Figure 4. Extended Kalman Filter Algorithm

Fig. 4 demonstrates the real-time recursive process of the extended Kalman filter algorithm.

### A. Time Update Model

#### 1) Attitude Strapdown Theory for a Quaternion:

In order to propagate the attitude in time, the quaternion kinematics equation is:

$$\dot{q}(t) = \Omega(\omega_t)q(t) \tag{8}$$

$$\text{Where } \Omega(\omega_t) = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{9}$$

$$q(t) = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T \tag{10}$$

$$\omega_t = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T \tag{11}$$

$q(t)$ is the quaternion that denotes the current attitude for the system state. $\omega_t$ are the current angular rates from the rate gyros for the system input. If $\Delta t$ is small enough, the state matrix can be derived by the Euler method: $\Delta t = t - t_0 \to 0$,

$$q(t+\Delta t) \approx (I_4 + \Delta t\Omega)q(t) \overset{\Delta}{=} f_{\Delta t}(q(t), \omega_{t+\Delta t}, \Delta t) \tag{12}$$

- *Error model for time update:*

Equation (12) defines the nonlinear system propagation for the state $q$ and input $\omega$ in time update. In order to obtain an extended Kalman filter with a capability of gyroscope bias separation, the sensor bias model is implemented in the sate matrix by error dynamics analysis [8], [9].

We define the state error of a gyro as $\delta\omega$:

$$\omega_{true} = \omega_{sensor} - \delta\omega \tag{13}$$

$$\delta\omega(t) = \Delta\omega_{bias}(t) + w(t) \tag{14}$$

Where $w(t)$ is a sensor's white noise, and $\Delta\omega(t)$ is the gyro bias which is considered as constant since $dt$ is small:

$$\Delta\omega(t+\Delta t) = \Delta\omega(t) \tag{15}$$

The propagation of the attitude state error, $\delta q$ can be obtained by partial differentiation of Equation (12):

$$\delta q(t+\Delta t) = \frac{\partial f_{\Delta t}(t)}{\partial \omega(t)}\delta\omega(t) + \frac{\partial f_{\Delta t}(t)}{\partial q(t)}\delta q(t) \tag{16}$$

When time step $dt$ and the previous $\delta q$ is small, we assume:

$\frac{\partial f_{\Delta t}(t)}{\partial q(t)}\delta q(t) \approx 0$. By Jacobean Linearization:

$$\frac{\partial f_{\Delta t}(t)}{\partial \omega(t)} = \frac{1}{2}\cdot\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}\Delta t \overset{\Delta}{=} -W_k\Delta t \tag{17}$$

From Equation (12) and Equation (16), the gyroscope bias can be separated from the system state:

$$q_k = f_{dt}(q_{k-1}, \omega_k, dt) - \delta q_k \tag{18}$$

Thus the Discrete Time Update is:

$$q_k = q_{k-1} + \Omega(\omega_k - \Delta\omega) \cdot q_{k-1} \cdot dt + W_k \delta\omega \tag{19}$$

In the state space representation,

$$x_{k+1} = A_k x_k + W_k w_k \tag{20}$$

Where $x_k = [q_k \quad \Delta\omega_k]^T$ (21)

$$A_k = \begin{bmatrix} [I_4 + \Omega(\omega_k - \Delta\omega_{k-1}) \cdot dt]_k & -\dfrac{\partial f_{\Delta t}(t)}{\partial \omega(t)} \\ 0 & I \end{bmatrix} \tag{22}$$

*B. Measurement Update Model*

After extended Kalman filter estimation, the spatial magnetic field disturbance becomes tolerable within one stroke and the Earth's magnetic field direction remains constant within the whiteboard. It can be utilized as reference for attitude in the measurement update. The three orthogonal magnetometers in the MAG-μIMU measure the geomagnetic field with respect to (WRT) the body frame. On the other hand, by coordinate transform using the propagated attitude, it can be estimated from the constant geomagnetic field WRT the Earth frame. Hence the difference between the magnetometer measurements and the transformed geomagnetic field is feedback in the measurement update of the extended Kalman filter to correct for the error in attitude propagation [10].

Vector $\vec{q}_b$, $\vec{q}_e$ are introduced to represent the geomagnetic field WRT the Earth frame and the magnetometer outputs, respectively. The two vectors are expanded into quaternions: $M_{body} = [\vec{q}_b \quad 0]^T, M_{earth} = [\vec{q}_e \quad 0]^T$.

If quaternion $n$ denotes the current attitude, by coordinate transform from Equation (6),

$$M_{body} = n^* \otimes M_{earth} \otimes n \tag{23}$$

Multiply the quaternion $n$ to both sides of Equation (23), we obtain:

$$n \otimes M_{body} - M_{earth} \otimes n = 0 \tag{24}$$

$$\begin{bmatrix} -[\vec{q}_b]_\times & \vec{q}_b \\ -\vec{q}_b^T & 0 \end{bmatrix} n - \begin{bmatrix} [\vec{q}_e]_\times & \vec{q}_e \\ -\vec{q}_e^T & 0 \end{bmatrix} n = 0 \tag{25}$$

Where $[\vec{q}]_\times$ is the cross product matrix:

$$[\vec{q}]_\times \overset{\Delta}{=} \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{26}$$

From Equation (25):

$$\begin{bmatrix} -[(\vec{q}_b + \vec{q}_e)]_\times & \vec{q}_b - \vec{q}_e \\ -(\vec{q}_b - \vec{q}_e)^T & 0 \end{bmatrix} n = 0 \tag{27}$$

$$C \overset{\Delta}{=} \frac{1}{2} \begin{bmatrix} -[(\vec{q}_b + \vec{q}_e)]\times & \vec{q}_b - \vec{q}_e \\ -(\vec{q}_b - \vec{q}_e)^T & 0 \end{bmatrix} \tag{28}$$

Thus, there is no requirement for the state $q$ to be a unit quaternion. Let $C$ be the measurement matrix. The measurement update of the extended Kalman Filter is:

$$y = Cq \tag{29}$$

- *Error Model for the Measurement Update*

According to error dynamics analysis [8], [9], Let:

$$\delta q_k = q_k - q_k^{'} \tag{30}$$

$$\delta y = \frac{\partial y}{\partial q_b} \delta q_b + \frac{\partial y}{\partial q_e} \delta q_e + \frac{\partial y}{\partial q} \delta q \tag{31}$$

Where $\dfrac{\partial y}{\partial q_{be}} \delta q_{be} = 0$, $\dfrac{\partial y}{\partial q_b} \delta q_b \approx 0$ when the previous attitude state error is small.

From Equation (29):

$$\frac{\partial y}{\partial q_b} = \frac{1}{2} \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{32}$$

Thus the Discrete Measurement Update is:

$$y_k = C_k q_k - V_k \delta q_k \tag{33}$$

Where $V_k = -\dfrac{\partial y}{\partial q_b}$ (34)

- *Summary*

A mathematical derivation method is introduced to derive an extended Kalman filter to minimize random noise and input bias error. The attitude calculation is totally based on a quaternion. As proved in Equation (27), the attitude quaternion $q$ does not need to be unified in iteration. Further, any reference field sensor, such as star sensors and accelerometers, or a combination, can be applied to this process model.

## V. EXPERIMENT RESULTS

In pervious work, the attitude extended Kalman filter is implemented using accelerometers for feedback. However, accelerometers are reliable in motionless applications but are undependable during motion. Shown in Fig. 5 and Fig. 6 below, the experimental data from the MAG-μIMU demonstrates that inertial accelerations interfere with gravitational accelerations,

which cannot be separated from each other. In this experiment, the pen was rotated from 0 degree to 45 degree. Before rotation and afterwards, the filter tracked quickly and accurately. However, during motion, the filter was affected by the accelerometers' Euler angles, which can be interpreted into two rotations because of the acceleration and deceleration.

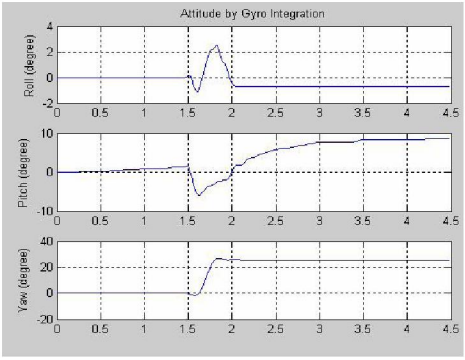


Figure 5. Accelerometers EKF: Attitude by Gyroscope Propagation

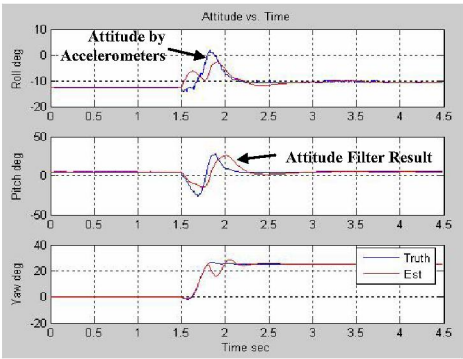


Figure 6. Accelerometers EKF: Filter Result & Accelerometer

Extensive simulation experiments are performed to check the convergence of the MAG Extended Kalman filter. The simulation software includes two parts: the sensor output generation and the real-time filtering. In order to generate the sensor output, the digital pen's physical properties and motion are modeled by the mass, inertia matrix, input forces and torques. The kinematics and dynamics module calculates the accelerations, angular rates and magnetic field strength under ideal conditions. The sensors' outputs are synthesized by aliasing the random Gaussian noise and constant sensor bias. For the filter part, the sensor outputs of angular rates and magnetic field strengths are processed by the extended Kalman filter in real-time. The attitude-in quaternion from the filter output is transformed into Euler angles for display.

Fig. 7 below shows the simulated sensor output. For rotational motion analysis, the input forces are set as [0, 0, 0]. In order to simulate a complex orientation, the torque vector in roll, pitch and yaw: [*L, M, N*] are set as:

$$L = 0.01 \cdot \sin(0.3t) \tag{35}$$

$$M = 0.01 \cdot \cos(0.3t) \tag{36}$$

$$N = 0.01 \tag{37}$$

The zero mean Gaussian noises are added to the ideal sensor outputs. The absolute maximum error amplitudes are 1.3 degree per second for the gyros and 0.04 Oersteds for the magnetometers respectfully. The initial attitude starts from 100 degrees in yaw angle. A constant sensor bias of 5 degrees per second is applied in yaw gyroscope output to verify the algorithm:

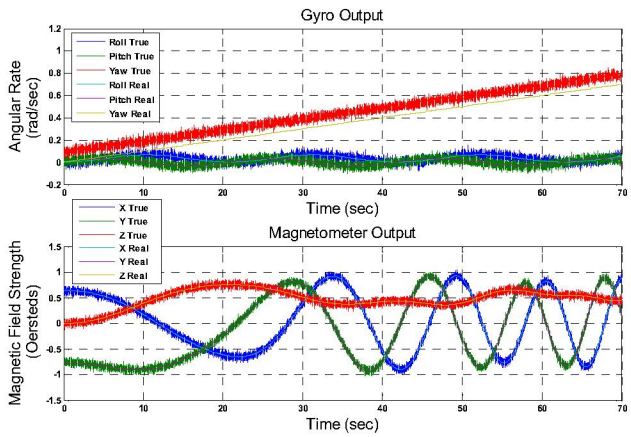


Figure 7. Synthetic Sensor Measurement

Fig. 8 illustrates the attitude result displayed in Euler angles. With the tracking ability of the extended Kalman filter, the initialization of the system state is simple. The attitude quaternion and gyro biases are set to zero. After iteration, the extended Kalman filter will estimate the gyro bias and remove it from the system state. According to magnetometer feedback, the filter's attitude estimation will converge and keep tracking automatically. The dash line in Fig. 8 shows the attitude propagated by the raw output from the gyroscopes. As shown in the figure, the random noise and bias error causes a large drift in the rolling, pitching and yawing compared with the filter output.

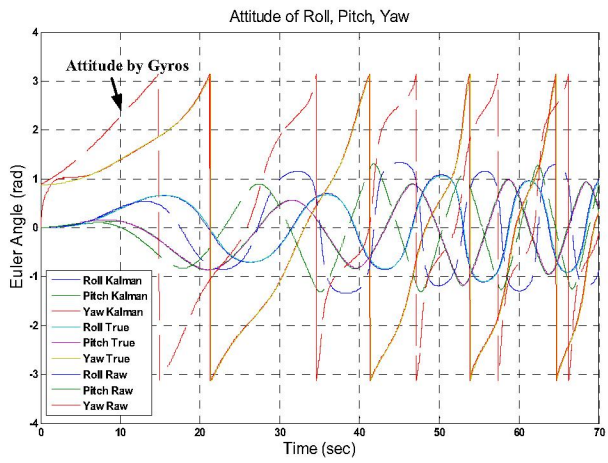


Figure 8. Simulation Attitude Comparison: Filter Result & Gyroscope Propagation

The extended Kalman filter was tested using real sensor measurements. The MAG-μIMU transmits the digital sensor measurements to the computer wirelessly via the Bluetooth

connection. The filter software in the computer processed the sensor data and calculated the attitude and sensor bias in real time.

The MAG-µIMU was held still for 4 seconds. Then continuous 90 degrees rotations were performed to test the tracking performance. The sensor module was rotated counterclockwise for 90 degrees and clockwise back to 0 degrees along the sensing axis of the roll gyroscope. At the end of the 7th rotation, the MAG-µIMU was suddenly held still again to test the convergence capability from dynamic input to static input.

Fig. 9, 10 below show the six raw sensor output and the estimated attitude in Euler angles. Within the first iteration, the estimated attitude converged according to the observations from the magnetometers. The dash lines show the attitude propagated by raw output from the gyroscopes. The sensor errors accumulated and caused the attitude drift.

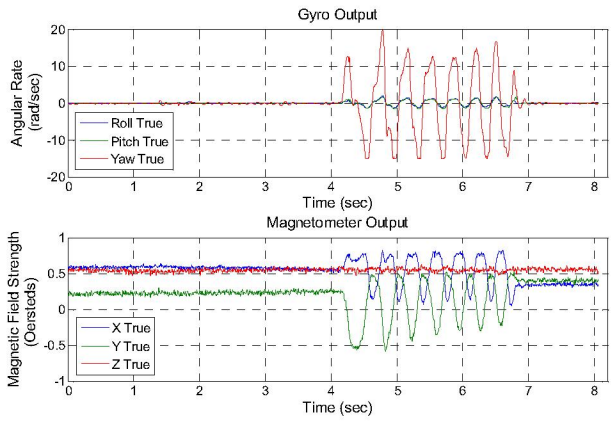


Figure 9. Real Senor Data from MAG-µIMU

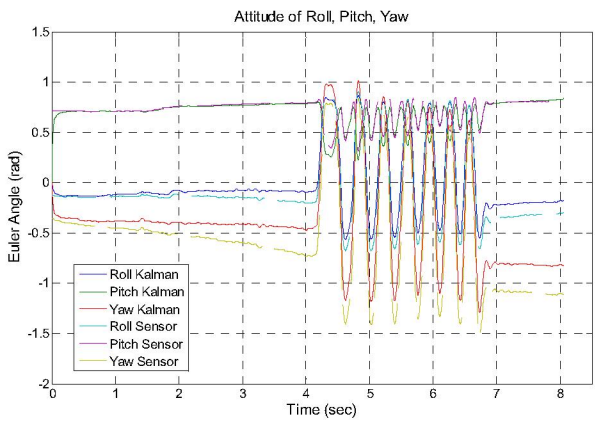


Figure 10. Experiment Attitude Comparison: Filter Result & Gyroscope Propagation

## VI. CONCLUSION

This paper presents a complete design of an extended Kalman filter for real-time attitude estimation of a moving rigid body. We also talked about a general approach to improve the Kalman filter to remove the input state's bias. The attitude representation is completely in quaternion, instead of Euler angles. The Euler angles have triangle functions which will cost more computation resources and cripple the filter due to the singularity problem. The measurement update is also applicable to other reference field sensors such as accelerometer filters. Further, no quaternion normalization is needed during filter iterations.

Extensive experiments were conducted to verify the convergence performance of the extended Kalman filter. The filter achieved good results for all tests using ideal simulation data and actual sensor data. The filter can track the pen attitude with the MAG-µIMU installed. The filter system is stable without the singularity or normalization problem even after consecutive, rapid 90 degree rotations.

However, the current filter has limitation in correcting the attitude error along the magnetic field direction. For future work, the Kalman filter will be expanded by the accelerometers in the measurement update. The implementation is similar to the magnetometer feedback. The attitude will be updated when the pen is static and the output from the accelerometers can be trusted as a gravity vector. Also, this long term correction will help eliminating magnetic field interference.

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## REFERENCES

[1] Guanglie Zhang, G. Y. Shi, Y. L. Luo, H. Wong, Wen J. Li, Philip H. W. Leong, and Ming Yiu Wong, "Towards an Ubiquitous Wireless Digital Writing Instrument Using MEMS Motion Sensing Technology," *Proceedings AIM '2005*, IEEE/ASME, 2005, pp. 795-800.

[2] Roger C. Hayward, Demoz Gebre-Egziabher, Matthew Schwall, J. David Powell and John Wilson. "Inertially Aided GPS Based Attitude Heading Reference System (AHRS) for General Aviation," *Proceedings of ION GPS '97*, Kansas City, MO, September, 1997.

[3] Eun-Seok Choi, W. Chang, W. C. Bang, J. Yang, S. J. Cho, J. K. Oh. J. K. Cho, and D.Y. Kim, "Development of the Gyro-free Handwriting Input Device based on Inertial Navigation System (INS) Theory," *Proceedings of SICE Annual Conference '2004*, SICE, 2004, pp. 1176-1181, vol. 2.

[4] D. H. Tittertoin, J. L. Weston, *Strapdown Inertial Navigation Technology, 2nd Edition*, AIAA, Reston, USA, 2004.

[5] Jack B. Kuipers, *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality*, Princeton University Press, Princeton, N. J. , 1999

[6] James R. Wertz, *Spacecraft Attitude Determination and Control*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[7] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. San Diego, CA: Academic, 1970.

[8] B. Friedland, "Treatment of Bias in Recursive Filtering," *IEEE Transactions on Automatic Control*, vol. AC-14, pp.359-367, 1969.

[9] Foxlin, E., "Inertial Head-Tracker Sensor Fusion by a Complementary Separate-Bias Kalman Filter," *Proceeding of VRAIS '96*, IEEE, 1996, pp. 185-194

[10] Markley, F. L., "Attitude Determination Using Vector Observations: a Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences*, Vol. 41, No. 2, Apr-Jun. 1993, pp. 261-280.